Line Integral

special case if $F = \nabla f$ is a gradient field

much easier!

Theorem $c : [a,b] \to \mathbb{R}^n$ a curve

$F = \nabla f$ gradient field

$$\Rightarrow \quad \int_c F \cdot ds = f(c(b)) - f(c(a))$$

Remark: we only need to know endpoints of curve!

<u>Example:</u> Let $F(x,y,z) = (2xyze^{x^2}, ze^{x^2}, ye^{x^2})$

Let C be a curve from $(0,0,0)$ to $(1,1,2)$

Calculate $\int_C F \cdot ds$

<u>Sol.</u> check: if $\boxed{f(x,y,z) = yze^{x^2}}$

$\Rightarrow F = \nabla f$

(e.g. $\frac{\partial f}{\partial x} = 2xyze^{x^2}$

Theorem $\Rightarrow \int_C F \cdot ds = f(1,1,2) - f(0,0,0)$

$= 2e - 0 = 2e$

$1 \cdot 2 \cdot e^{1^2}$ -

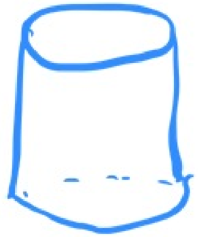
Not every vector field is a gradient field

7.3 Parametrized surfaces

so far: surfaces given as graphs of functions $f: D \subset \mathbb{R}^2 \to \mathbb{R}$

$$z = f(x,y)$$

many surfaces can not be described that way!

Examples: (a) cylinder  (open can w/o bottom)

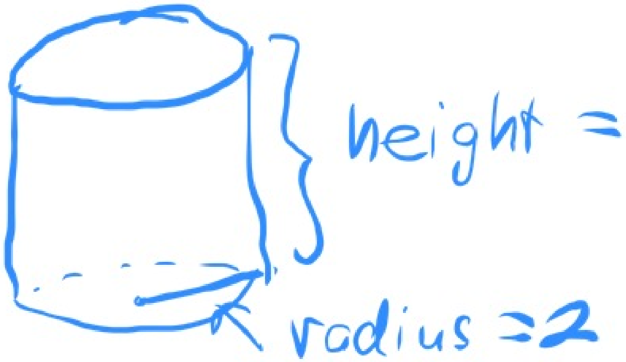
(b) sphere 

need notion of <u>parametrized surface</u>

<u>Def</u>: A parametrized surface is a map $\Phi : D \subset \mathbb{R}^2 \longrightarrow \mathbb{R}^3$

<u>Examples</u>:

ⓐ cylinder



Use cylindrical coordinates

cylinder given by $\boxed{x = 2\cos\theta, \quad y = 2\sin\theta, \quad z = z}$ describes Φ

$$\boxed{\begin{array}{c} 0 \le \theta \le 2\pi \\ 0 \le z \le 3 \end{array}} \Rightarrow \text{describes } D = [0, 2\pi] \times [0, 3]$$

more precisely

$$\Phi : [0, 2\pi] \times [0, 3] \longrightarrow \mathbb{R}^3$$

$$\Phi(\theta, z) = (2\cos\theta, 2\sin\theta, z)$$

2 sphere of radius 2

use spherical coordinates

$$\rho = 2$$

$$x = 2\sin\phi\cos\theta$$

$$y = 2\sin\phi\sin\theta$$

$$z = 2\cos\phi$$

$$\boxed{\begin{array}{l} 0 \le \phi \le \pi \\ 0 \le \theta \le 2\pi \end{array}}$$

describes $D = [0, \pi] \times [0, 2\pi]$

$$\Rightarrow \Phi(\phi, \theta) = (2\sin\phi\cos\theta, 2\sin\phi\sin\theta, 2\cos\phi)$$

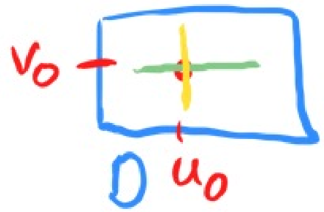
Remark: This is essentially the same way as we describe a position on Earth via latitude ($\sim \phi$) and longitude ($\sim \theta$)

for Earth: $-\frac{\pi}{2} \le \phi \le \frac{\pi}{2}$

(negative ϕ $\sim$ southern latitudes)

Tangent vectors for parametrized surfaces

Φ :



$\longrightarrow$

use coordinates u, v for Φ

green line = (u, v_0) u varies

$\longrightarrow$ mapped to curve $\Phi(u, v_0)$

$\Rightarrow$ get tangent vector $T_u = \partial/\partial u \, \Phi(u_0, v_0)$

$$T_v = \frac{\partial}{\partial v} \Phi(u_0, v_0)$$

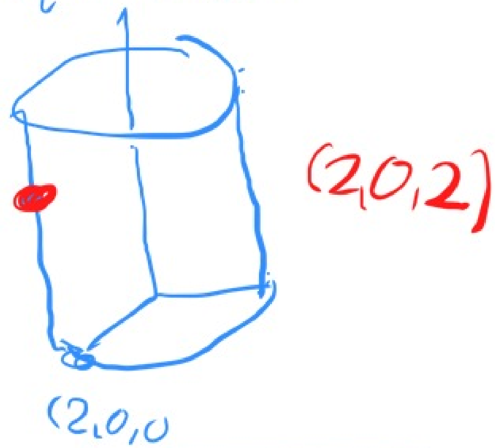
Result: If $\Phi : D \to \mathbb{R}^3$ is a parametrized surface

then we obtain two tangent vectors at $\Phi(u_0, v_0)$

$$T_u = \partial/\partial u \, \Phi(u_0, v_0)$$

$$T_v = \partial/\partial v \, \Phi(u_0, v_0)$$

Example: Calculate T_θ and T_z for the cylinder at point $(2,0,2)$



$$\boxed{\Phi(\theta, z) = (2\cos\theta, 2\sin\theta, z)}$$

$$(2,0,2) = \Phi(0,2)$$

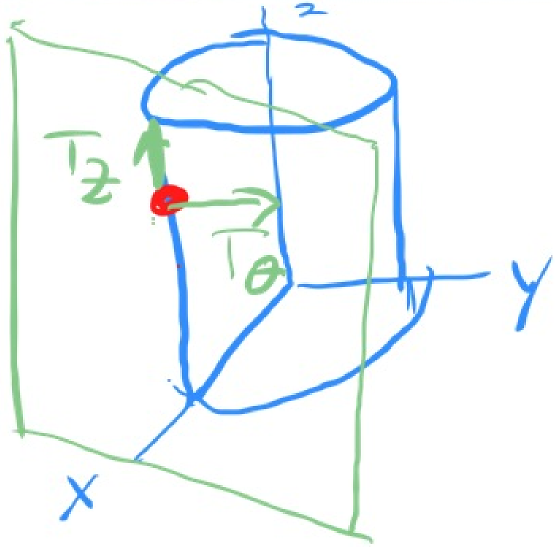
Sol. $\dfrac{\partial \Phi}{\partial \theta} = (-2\sin\theta, 2\cos\theta, 0) \Rightarrow$

$$\boxed{T_\theta = \frac{\partial \Phi}{\partial \theta}(0,2) = (0,2,0)}$$

$$\Phi(\theta, z) = (2\cos\theta, 2\sin\theta, z)$$

$$\frac{\partial \Phi}{\partial z}(\theta, z) = (0, 0, 1)$$

$$\Rightarrow \boxed{T_z = \frac{\partial \Phi}{\partial z}(0, 2) = (0,0,1)}$$



$(2,0,2)$

next time:
tangent plane
given by
$x = 2$

here: tangent plane at cylinder

= plane spanned by T_z and T_θ